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KEY

CONTAINING THE

ANSWERS TO THE EXAMPLES

IN THE

INTRODUCTION TO ALGEBRA

UPON THE

INDUCTIVE METHOD OF INSTRUCTION.

BY WARREN COLBURN, A. M.

STEREOTYPED AT THE BOSTON TYPE AND STEREOTYPE FOUNDRY.

BOSTON:

PUBLISHED BY HILLIARD, GRAY & CO.

1835.

KEY.

I.

1 & 2. See Book.

3. Saddle $13, horse $117.

4. A's share $82, B's $246, C's $328.

5. The first 6, the second 12, the third 18, and the fourth 24 cents.

6. The share of the widow was 4000 crowns; that of the son 2000, and that of the daughter 1000.

7. A paid $317; B $951; C $1268; and D $2219.

8 & 9. See Book.

10. 12 of each sort.

11. 30 days.

12. The first 13 days, the second 39, and the third 78.

13. 3 oxen, 9 cows, and 18 calves.

14. 22 barrels.

15. 11 days.

II.

1 & 2. See Book.

3. $104.

4. See Book.

5. 80 trees.

6. 96 sheep.

7. $160.

8. 39 geese.

9. $120.

10. 72 bushels.

11. $102\frac{6}{7}$ loads.

12. 84 years.

13. £11. 9s. 9.1$\frac{37}{47}$.
14. 80 scholars.
15. 72 lb.
16. 40 inches.
17. A's 15, B's 39 years.
18. See Book.
19. The first $120, the second $80, and the third $90.
20. 48$\frac{8}{29}$ bushels of oats, 34$\frac{14}{29}$ bu. barley, 17$\frac{7}{29}$ bu. rye.
21. The first has 24, the second 16.
22. They received £56, £35, and £7, respectively.
23. $720, and $560.
24. $192, $448, $320, respectively.
25. Their sums were $140, $90, $190, respectively.

III.

1. See Book.
2. Horse $164, and chaise $136.
3. Son's share $7175, daughter's share $4825.
4. See Book.
5. 35, 50, and 74 yards, respectively.
6. The wife had $14250, the elder son $16250, the younger son $11250, the daughters each $7750.
7. The numbers were 439 and 504.
8. They received 12, 10, 7, and 3 shillings, respectively.
9. Brandy 29, wine 44, and water 73 gallons.
10. See Book.
11. 22 years.
12. He had 6 shillings, and there were 14 beggars.
13. Their ages were 10, 14, 18, 22, 26, and 30 years.
14. £14, £24, £38, respectively.
15. 48 pounds.
16. See Book.
17. The whole fortune was 28800 crowns. The shares were 13400, 8800, and 6600 crowns, respectively.
18. The whole fortune was 12000 livres. The shares were 3000 livres each.

19. 128 lb. copper, 84 lb. tin, and 76 lb. lead
20. 24000 men.
21. 80.
22. 60 years.
23. A £27, and B £12.
24. 85 gallons of wine and 35 of cider.

IV.

1. See Book.
2. Their ages were 36 and 54 years.
3. £24, £60, and £192, respectively.
4. See Book.
5. 3 cents apiece for the pears, and 5 for the oranges
6. 6 cents for oranges, and 3 cents for lemons.

7&8. See Book.

9. The man's age 45 years, and the wife's 15.
10. 147 sheep.
11. 20 days.
12. A lost £96, and B £48.
13. 42 shillings.
14. A had $20, and B $4.
15. A $2440, and B $980.
16. $10.
17. 3, 9, and 29, respectively.
18. 9, 11, and 14 crowns, respectively.
19. Flour $4, and butter $6.
20. Wine $1.60, and brandy $1.85 per gallon.
21. Oxen $43, cows $25.
22. 10 of each sort.
23. See Book.
24. They had 13 and 22 guineas respectively.
25. 18 barrels at $8, and 27 barrels at $5.
26. 5 oxen and 10 cows.
27. See Book.

28. The less 82, and the greater 115.
29. See Book.

V.

1, 2, & 3. See Book.
4. Horse $100, chaise $150.
5. Their ages were 18 and 28 years.
6. 150 and 180 miles.
7. Saddle $12, and horse $126.
8. Hare 280 rods, greyhound 320.
9. $26, $10, $19, and $81, respectively.
10. 15 barrels of beer, and 21 of cider.
11. The first £62. 4s. 5d. $1\frac{1}{3}$qrs.; the second £22. 4s. $5\frac{1}{3}$d.
12. Man 24 years, wife 16.
13. See Book.
14. 20 cards.
15. $63, and $120, respectively.
16. 8 bushels of wheat, and 12 of rye.
17. 65.
18. Time past 54 years, and 45 to come.
19. 20 and 30.
20. $38.
21. Man's age 56 years, and wife's 40.

VI.

1 & 2. See Book.
3. A began with $410, and B with $205.
4. A $$6\frac{1}{2}$, and B $$13\frac{1}{3}$.
5. 18 and 36.
6. See Book.
7. Horse $152, chaise $189.
8. A had $45, and B $75.
9. $126.

VII.

1. A cut off 48, and B 28.

2. 13 deals.
3. 16 and 24.
4. A's share $450, B's $270.
5. Greater $108, less $60.
6. A paid $10.805; B $11.148; C $10.587; and D $10.46.
7. Cover 20 oz., second cup 16 oz.
8. 36 passengers.
9. From A to B 72 miles; from B to C 14; from C to D 48 miles.
10. $4.80.
11. They paid $60, $50, and $30, respectively.
12. The men received £25, and the women £21.
13. 2000 sheep.
14. A put in $215, B $145, and C $305.
15. 53 and 32.
16. 43 and 24.
17. Sugar 20 cents per pound, lemons 3 cents each.
18. Oranges 6 cents, and lemons 4 cents each.
19. $24.
20. 120 bushels.
21. $1400.
22. $11\frac{1}{4}$ pence.
23. The parts are 28 and 32.
24. The parts are 50 and 35.
25. The parts are 24 and 76.
26. The parts are 32 and 16.
27. 8 received 6 pence apiece, and 12 received 16 pence apiece.
28. 60 guineas and 148 crowns.
29. 27 in the less, and 81 in the greater.
30. Barley 45 pence, and oats 30 pence per bushel.
31. 120 of each sort.
32. $2\frac{1}{12}$ hours.
33. $2\frac{6}{17}$ days.

34. 30 hours.
35. 20 days.
36. The less 13, and the greater 26 gallons.
37. Their annual income is £125; A spends £100, and B £180.
38. 120 guineas.
39. 50 each.
40. The parts are 12, 8, and 6.
41. The parts are 12, 18, and 24.
42. £15 and £20 respectively.
43. \$384, and \$441 respectively.

VIII.

1&2. See Book.
3. Quinces 3 cents, and melons 4 cents each.
4. Barley 3 shillings, and oats 2 shillings per bushel.
5. Shoes \$2, and boots \$6.
6. Broadcloth 15 shillings, and taffeta 3 shillings.
7. Men 5 shillings, and boys 2 shillings, 6 pence.
8. The Port was £3, and the Sherry £2 per doz.
9. Horse \$210, and chaise \$240.
10. A had 5, and B 7 sheep.
11. A has \$13, and B \$11.
12. 15 bushels of rye, and 12 bushels of wheat.
13. 126 gallons of wine, and 42 of gin.
14. See Book.
15. 40 pears, and 60 apples.
16. 50 at 2 for a cent, and 60 at 3 for 2 cents.
17. The numbers are 24 and 18.
18. The numbers are 15 and $10\frac{1}{2}$.
19. A's age was 44 years, and B's 20.
20. The upper part 10 feet, and the lower 14.
21. $8\frac{4}{7}$ gallons of the better, and $11\frac{3}{7}$ of the worse.
22. Smaller 32 gallons, larger 68 gallons.
23. 33 sheep, and 20 cows.

24. 10 bushels of each.

25. 36 galls. at 7 shillings, and 42 galls. at 10 shillings.

IX.

1 to 13. See Book.

14. $p = \frac{i}{t\,r}$.

15. $r = \frac{a - p}{t\,p}$.

16. $t = \frac{a - p}{r\,p}$.

17. See Book.

X.

1 & 2. See Book.

3. $72\,e\,g\,h$.

4. $91\,a\,a\,a\,c\,c\,d$.

5. $455\,a\,a\,b\,b\,b\,c\,d$.

6. $690\,a\,c\,d$.

7. $275\,a\,b\,x\,x$.

8. $504\,a\,x\,x\,y\,y\,y$.

9. See Book.

10. $a^4\,b^6\,c^3$.

11. $6\,a^6\,b^5\,c^3\,d^3$.

12. $a^{10}\,b^2\,c^3$.

13. $35\,a^6\,b\,c\,x^5\,y^4$.

14. $68\,b^7\,c\,d^4\,e^3$.

15. $46\,a^4\,b\,x^{10}$.

16. $108\,a^5\,x\,y^4$.

XI.

1. $8\,a\,b - 2\,a^2\,m - 3\,a\,m$.

2. $17\,a\,n^2 - 6\,m - 2\,x^2 + 7\,b\,m - 12\,y + 5\,a\,x^2$.

3. $- 6\,a\,m\,b + 32 - 29\,m\,y + 19\,a\,c\,b + 41\,a\,m\,y - 3\,b\,n + 3\,n\,x$.

4. $-xy+2ax-2ay-3axy-33+37arx-11am.$

5. $13ax+12bx+46+27byx+48acd-x-15a.$

XII.

1 & 2. See Book.

3. $15a^2y+13ay^2+10a-8+b.$

4. $44axy-21ax+5.$

5. $110-21y+16ab.$

6. $2a-2.$

7. $5abz.$

XIII.

1 & 2. See Book.

3. $18a^3b^4+3ab^5c^3.$

4. $7a^2b^4c^3d^5+364a^4b^5c+91a^2b^8c^2d.$

5. $6a^2b^2dx^2+12a^2b^2x^4.$

6. $13a^2b^2x^6+39a^2b^3x^5.$

7 & 8. See Book.

9. $6ab^2d-2cd.$

10. $10a^2bd+5ab^2d-15abc.$

11. $15abc^2d-5acef-10a^2c^2.$

12. $8a^6b^6c-20a^5b^5+4a^3b^7.$

13. $17a^4c^2d^2-a^3cd+5a^5cdx-a^4b^2cdx.$

14, 15, & 16. See Book.

17. $10a^2d^2+6a^2cd^2+6a^3c^2d-10a^4c^2.$

18. $65a^2cry^4+11abcy^5+15c^2y^6+91a^3bry^2-14a^2b^2y^4+39a^2ry-6aby^2+9cy^3.$

19. $25a^3c^3+2a^4c^2-8a^5c+11a^2c^4.$

20. $a^3-a^2c-ac^2+c^3.$

21. $6a^8-6a^2b^4+6a^4b^2-6b^6.$

22. See Book.

23. $8a^2bd+12ab^2c+8ab-4adc-6bc^2-4c.$

24. $12a^4b^2+4a^3b^3-6a^2b^3-2ab^4-6a^2b-2ab^4.$

25, 26, & 27. See Book.

28. $28m^2 - mn - 15n^2$.
29. $a^3 - 2ay^2 + y^3$.
30. $n^3 - x^3$.
31. $a^4 + a^2b^2 + b^4$.
32. $10x^3 - 15x^2y + 20xy^2 - 12x^3y + 14x^2y^2 - 18xy^3 - 8y^4$.
33. $6a^5c^2 - 22a^4c^3 + 24a^3c^4 - 8a^2c^5 - 21a^3c^3 + 35a^2c^4 - 14ac^5$.
34. $6a^4 - 5a^3x + 6a + a^2x^2 - 2x - 6a^3 + 3a^2x - 6$.
35. $21a^4b + 6a^2b^2 - 3a^2 - 14a^2b^3 - 4b^4 - 7a^2b + 1$

XIV

1 & 2. See Book.
3. $4abc$.
4. $4bc$.
5. $2a$.
6. $3bc$.
7. $23c$.
8. 17.
9. $4a$.
10. See Book
11. $35bd^2$.
12. $4a^2$.
13. $3x^2y$.
14. $3a^2x^2$.
15. $6r^2m^2$.
16. p^2y^2.
17. 73.
18. $120art^2$.
19 & 20. See Book.
21. $c - 5bd + 3a^2d^2$.
22. $3ab - 6a^3c + 5ad$.
23. $4a^7b^2c$.
24. $42a^7$.

XV.

1, 2, & 3. See Book.

4. $\frac{10a^2bc + abc^2 - 3bc^2}{3ab}$.

5. $\frac{25a^2bc - 10abm^2 - 15acn + 6nm^2}{13ac}$.

6. $\frac{32amx^2 - 48ax^3 - 6bmx + 9bx^2}{2a + 7x^2}$.

XVI.

1 to 5. See Book.

6. $\frac{2b}{d}$.

7. $\frac{2ab^2}{cd}$.

8. $\frac{a^3b}{3dm}$.

9. $\frac{3}{5ac}$.

10. $\frac{7bc}{3a}$.

11. $\frac{4ac}{5bd}$.

12. $\frac{17acd}{2bm^2nr}$.

13. $\frac{13ab}{15c^3d^3}$.

14. $\frac{27mr}{56a^3c^3bd}$.

15. $\frac{3a - 2b}{6abcd}$.

16. $\dfrac{7am - 13bc}{6a^2bd - 15ab^3}$.

17. $\dfrac{12acd}{35abn^4 - 7bmn^6}$.

18. $\dfrac{17c - 3x^2}{8a^4n - 22a^2n^2 - 21n^3}$.

19. $\dfrac{37ad}{184b + 12bx - 138x - 9x^2}$.

20. $\dfrac{2a - d}{21a^2 - 16acd + 4a - 16c^2d^2 + 8cd - 1}$.

21, 22, & 23. See Book.

24. $\dfrac{9a^2dm}{20bc^3n}$.

25. $\dfrac{36a^2nx^3}{26b^2ry^2}$.

26. $\dfrac{16a^2bfm}{15cd^2m}$.

27. $\dfrac{28b^3dm}{117anx^3}$.

28. $\dfrac{4a^2c - 2abc}{15ab^2 + 5abc}$.

29. $\dfrac{10a^2m^4 - 15a^3m^3}{8a^2cm + 4acm - 20ac^2 - 10c^2}$.

30. $\dfrac{6a^2bc + 9ab - 2ac^2 - 3c}{25abc^2 - 10acd^2 - 10abcd + 4ad^3}$.

31. $\dfrac{26a^2c - 13acm + 39acm^2}{119a^2m^5 - 7acm^3 + 35am^3}$.

32. See Book.

33. $\dfrac{a}{c}$.

34. $\dfrac{3a}{3b}$, or $\dfrac{a}{b}$.

35. $\frac{17ab}{4c}$.

36. $\frac{16}{6am}$, or $\frac{8}{3am}$.

37. $\frac{ab}{5m^4}$.

38. $\frac{3}{a}$.

39. $\frac{7}{8}$.

40. $\frac{3ac}{ab-c}$.

41. $\frac{17-4bc}{4-3ab-1}$.

42. $\frac{23m-13}{5d-m^3+6m^2ac^2}$

43 & 44. See Book.

45. $7acm$.

46. 25.

47. $18x$.

48. $12m^2y^2$.

49. $13ab-m$.

50. $15ac+37bc$.

51. $47am^2+3b-c$.

XVII.

1, 2, & 3. See Book.

4. $\frac{2m^2y^2}{br}$.

5 $\frac{4b^3x}{a^2}$

6. $\dfrac{3\,a\,c^2 - 5\,a^5}{b\,c + 11 a^3\,b}$.

7. $\dfrac{3\,m^5 - 6\,x^2}{12\,a\,x + 9 - 10\,m^2\,x^2}$.

8 & 9. See Book.

10. $\dfrac{8\,r^2}{b^2 n\,y}$.

11. $\dfrac{3\,m\;n\;r^3}{2\,b\,y^2}$.

12. $\dfrac{2\,c^2\,d\,x^2}{7\,m^4\,r}$.

13. $\dfrac{23\,y^3}{3}$.

14. $\dfrac{8\,a^2\,c^5}{3\,m\,r^6}$.

15. $\dfrac{17}{13\,c^2\,x^3}$.

16. $\dfrac{2\,c}{a y}$.

17. $\dfrac{2\,m^2}{3\,a^3\,y^2}$.

18. $\dfrac{15\,a^4\,b}{7 c^2 y^3\,x}$.

19. $\dfrac{a + b}{2\,c - d}$.

20. $\dfrac{2 - 7\,a^3 b + 15\,d}{13\,a^5\,d}$.

21. $\dfrac{3\,m - 9\,a^3 + 7\,a\,m^2}{5\,a^3\,m\,d - 2\,c\,m}$.

22. $\dfrac{13\,a\,c + b\,c}{m^2 - c}$.

23. $\frac{3}{2\,b\,c}$.

24. $\frac{3\,c^2}{a\,b^2 - b^2\,d}$.

XVIII.

1 to 9. See Book.

10. $\frac{4\,a\,c\,d\,g + 3\,b^2\,e\,f}{2\,b\,d\,e\,g}$.

11. $\frac{15\,a\,b\,h\,m + 2\,r\,e\,c\,h + 6\,e\,b\,n\,r}{6\,r\,b\,h}$.

12. $\frac{9\,a\,p + 6\,b\,n}{6\,m\,n\,p}$.

13. $\frac{2\,a\,r\,m\,n + 4\,c\,m^3 + 15\,b\,c\,d}{10\,b\,m^3\,n}$.

14. $\frac{9\,m^3\,s + 2\,a\,n^2\,r^2}{3\,m\,n^5\,r^2}$.

15. $\frac{15\,a\,h^3\,n^3\,x + 2\,b\,c\,n\,x + 8\,h\,x\,m^3\,r}{4\,h^3\,n^3\,x}$.

16. $\frac{2\,a\,c + 39\,b\,c\,d}{3\,b}$.

17. $\frac{m + 4\,a\,c\,n - 10\,b\,n}{2\,n}$.

18. $\frac{13\,a^2\,n^2 - 4\,c^2 + 22\,a^3\,c\,n}{2\,a^2\,n}$.

19. $\frac{59\,a^2\,m\,c - 6\,b}{21\,b\,m^2}$.

20. $\frac{27\,a^2\,b - 2\,c + 104\,a^3\,b}{4\,a\,b + 32\,a^2\,b}$

21. See Book.

22. $\frac{15\,d\,m\,n - 4\,a\,b\,c\,x}{6\,c^3\,d\,x}$.

23. $\frac{b^2 c d e - 9 c d m x}{21 b^3 m^3 x^3}$

24. $\frac{3 n r x^4 - 2 b d m y^2}{2 m n x^3 y^5}$.

25. $\frac{4 m^3 s r y - 15 \quad t^2 x y}{6 m^5 s^2 t^3}$.

26. $\frac{15 x y - 17 r x^2 m b^3}{3 m^2 b^3}$.

27. $\frac{77 n^3 r s - 39 m^2 y x^2}{21 n^5 x^3}$.

28. $\frac{26 a b c m + 2 b^2 c m - 3 a c}{2 b m}$

29. $\frac{a^2 x - 98}{14 a^2 m}$.

30. See Book.

31. $\frac{5 m n x^3 - 26 a d n x^2 - 34 a d}{12 a d m x}$.

32. $\frac{15 a d n x^2 + 55 a n}{12 d^2 x^3 - 20 d x}$.

XIX.

1 to 7. See Book.

8. $\frac{15 a b}{2}$.

9. $\frac{13 a c}{b}$.

10. $\frac{17 a b m}{2 c}$

11. $\frac{2 a^2 t}{3 b}$.

12. $\frac{9 x^3 y}{2 a m}$.

13. $\frac{10\,a\,c^2 - 5\,b\,c^2}{3\,a}$.

14. $\frac{119\,a^3\,c\,x - 14\,a^2\,b\,c + 7\,a^2\,c^2}{13\,a\,b - 2}$.

15. $\frac{77\,a^2\,c\,x^3 - 21\,a\,c\,x - 33\,a^2\,c\,x^2 + 9\,a\,c}{2}$.

16. $\frac{m\,b}{3\,a\,d}$.

17. $\frac{5\,c\,d^3}{3\,x\,y^3}$.

18. $\frac{119\,m\;y^4}{15\,a^2\;n^2\,x^2}$.

19. $\frac{84\,a^4\,b^2\,m}{175\,n^4\,5y}$.

20. $\frac{39\,a\,c\,x + 6\,b\,c^2\,x}{52\,a\,b - 8\,a\,x^2 + 28}$.

21. $\frac{4\,a^2 - 9\,c^2\,d^2}{4\,a^2\,m^2 - 25\,a^2\,x^2}$.

22. $\frac{27\,d\,m\,x - 10\,d^2 - 5\,m^2\,x^2}{39\,a\,m\,y^2 + 39\,a\,d\,m\,y}$.

XX.

1. $a + x$.
2. $a - b$.
3. $b^2 + x$.
4. $x - y$.
5. $x^2 - x\,y + y^2 - \frac{2\,y^3}{x + y}$.
6. $5\,a + 4\,b$.
7. $x^2 + x\,y - y^2$

8. $a - 3 + \frac{54 - 27\,a}{a^2 - 6\,a + 9}$.

9. $2\,a^2 + \frac{-7\,a^2 - 6\,a^2\,x + 6\,a - 3}{2\,a^2 + 3\,x - 1}$.

10. $a + x$.

11. $2\,x^3 + 4\,x^2 + 8\,x + 16$.

12. $2\,a + \frac{a\,b}{2\,a - b}$.

13. $2\,a^2 + 2\,a + 5$.

XXI.

1. See Book.

2. $x = \frac{26\,a\,c - 2\,c\,d}{4\,c^2 + b}$.

3. $x = \frac{a\,b^2 - 2\,a - 4\,b^2\,c}{3\,a\,b - 12\,b\,c}$.

4. $x = \frac{-54\,a\,b^2\,c + 26\,a\,b\,c + 2\,a\,d}{14\,b - 7 - 135\,b^2\,c + 65\,b\,c + 5\,d}$.

5. $x = \frac{6\,a\,b\,c + 6\,b^2\,c - 30\,a\,b^3\,c}{4\,a^2 - 3\,a\,b + 15\,a\,b^3}$.

XXII.

1. $5000.
2. £30.
3. £48 at 5 per cent. £50 at 6 per cent.
4. 127 sheep.
5. 24 remaining, and 8 lost.
6. Length 30 rods, breadth 20.
7. 2 and 10.
8. See Book.
9. 12 persons, who paid 5 shillings each.
10. There are 9 persons, and they receive $300 each.
11. $\frac{4}{15}$.
12. See Book.

13. Numerator $\dfrac{a\,p\,(m+n)}{n\,p-m\,q}$, denominator $\dfrac{a\,n\,(p+q)}{n\,p-m\,q}$.

14. $\frac{2}{4}\frac{7}{3}\frac{3}{2}$.

XXIII.

1 & 2. See Book.

3. A's age 48, B's 33, C's 15.

4. A had 10, B 20, C 30.

5. The parts are 5, 15, 2, and 50.

6. A at \$1.00, B at \$1.20, C at \$1.40, and D at \$2.00, per gallon.

7. $x = 4, y = 3, u = 8, z = 14$.

8. A's money $\dfrac{5\,d}{17}$, B's $\dfrac{11\,d}{17}$, C's $\dfrac{13\,d}{17}$.

XXIV.

1 & 2. See Book.

3. The equations by the conditions are as follows:

Let $x =$ the daily wages of the man,
$y =$ " " his wife and son.

Then
$$15\,x + 9\,y = 14.25.$$
$$12\,x + 5\,y = 13.50.$$

If we reduce these equations, we find

$$x = \$1.52\tfrac{3}{11}, \text{ and } y = -\$.95\tfrac{5}{11}.$$

The y being negative shows that there was some wrong supposition, or that it should enter into the equation with the contrary sign, thus

$$15\,x - 9\,y = 14.25.$$
$$12\,x - 5\,y = 13.50.$$

If these equations be reduced, the values will be the same as before, but both will be positive.

If now we translate these equations into ordinary language, it will show how the question should have been enunciated at first:

A labourer wrought for a man 15 days, &c. How much did he receive per day for himself, and how much did he pay back for the expense of his wife and son?

4. Let $x =$ man's daily wages,

$y =$ sum paid per day for his wife.

Then $$11\,x - 4\,y = 17.82.$$
$$23\,x - 13\,y = 38.78.$$

These equations being reduced, the values are

$$x = \$1.50\tfrac{4}{51},\text{ and } y = -\$0.32\tfrac{40}{51}.$$

If y be put into the original equation with the contrary sign, it will appear that the question should have been asked: how much did he receive per day for his wife?

5. Let $x =$ the daily wages of the labourer,

$y =$ " " his wife,

$z =$ " " his son.

Then $$7\,x + 4\,y + 3\,z = 7.89.$$
$$10\,x + 7\,y + 5\,z = 11.65.$$
$$8\,x + 5\,y + 8\,z = 7.54.$$

Reducing these equations, we find

$$x = \$1.00,\ y = \$.50,\text{ and } z = -\$.37.$$

Putting z into the equations with the opposite sign, it will appear that the son was an expense to the father, instead of earning something, and the question should have been so proposed.

6. See Book.

7. Let $x =$ the number.

Then by the conditions

$$\frac{7\,x}{11} - \frac{5\,x}{7} = 18$$

$$x = -231.$$

If we put $-x$ instead of x into the first equation, it becomes

$$-\frac{7\,x}{11} + \frac{5\,x}{7} = 18$$

or $$\frac{5\,x}{7} - \frac{7\,x}{11} = 18.$$

Which shows that the question should have been proposed thus:

What number is that, $\frac{5}{7}$ of which exceeds $\frac{7}{11}$ of it by 18?

8. Let $x =$ the numerator, and

$y =$ the denominator.

Then by the conditions,

$$\frac{x+1}{y}=\frac{3}{5}$$

$$\frac{x}{y+1}=\frac{5}{7}.$$

Reducing this, we find $x=-10$ and $y=-15$. Consequently both entered wrong into the conditions. Put both into the original equations with the contrary sign:

$$\frac{-x+1}{-y}=\frac{3}{5}$$

$$\frac{-x}{-y+1}=\frac{5}{7}.$$

Change the signs of both numerator and denominator in the first members. This will not alter the values* or the signs of the fractions themselves; consequently it will not be necessary to alter the signs of the second members.

$$\frac{x-1}{y}=\frac{3}{5}$$

$$\frac{x}{y-1}=\frac{5}{7}.$$

These will give positive values of x and y, and they show that the question should have been expressed thus:

What fraction is that, from the numerator of which if 1 be subtracted, its value will be $\frac{3}{5}$, but if 1 be subtracted from the denominator, its value will be $\frac{5}{7}$?

9. Let $x=$ the numerator, and
$y=$ the denominator.

Then by the conditions,

$$\frac{x-2}{y}=\frac{14}{13}$$

$$\frac{x}{y-2}=\frac{4}{5}.$$

* See Book, Art. XX., respecting the signs in division. The same principles apply to fractions, the denominator of which is a divisor, and the numerator a dividend. Every fraction has three signs, thus, $+\frac{-a}{+b}$, any two of which may be changed without altering the value.

These being reduced, we find $x = -12$, and $y = -13$. The substitutions being made in this as in the last, and the signs changed in the same manner, it will appear that the 2 should have been *added* to the numerator in the first equation, and to the denominator in the second.

10. Let $x =$ the greater number, and
$y =$ the less.

Then by the conditions,

$$x + y = 20$$
$$3x + 5y = 125.$$

Reducing these, we find $x = -12\frac{1}{2}$ and $y = 32\frac{1}{2}$; substituting $-x$ in the original equations, they become

$$y - x = 20$$
$$5y - 3x = 125.$$

Here it appears, first, that the difference of the numbers, instead of their sum, makes 20. Secondly, $y - x$, being a positive number, shows that y and not x is the larger number. Thirdly, the second equation shows that the difference between five times the greater and three times the less, is 125.

11. Let $x =$ the greater, and
$y =$ the less.

Then by the conditions,

$$x + y = 25$$
$$7x + 5y = 215.$$

These being reduced, we find

$$x = 45 \text{ and } y = -20.$$

Which shows that the equations should be in this form

$$x - y = 25$$
$$7x - 5y = 215.$$

The question, according to this, should be enunciated thus: It is required to find two numbers whose difference is 25, and, if the larger be multiplied by 7, and the smaller by 5, the difference of the products shall be 215.

XXV.

1. See Book.
2. $9a^3cd^2$.

3. $10\, a^{-1}\, c^4$.

4. $\frac{39}{2}\, a^{-2}\, b\, c^3\, d$.

5. $6\, a^2\, (b + d)^{-1}$.

6. $6\, a\, b\, c$.

7. See Book.

8. $5\, a^{-1}\, c\, (2\, b - c)^{-4}$.

9. $12\, a\, c^{-4}\, d^{-2}\, e$.

10. See Book.

11. $\frac{7}{2}\, a^2\, b^{-1}\, c^{-1}\, m\, (b\, c - d)^{-2}$.

12. $\frac{b\, c\, (b - 2\, c)^4}{9\, a}$.

13. $\frac{b\, c - 2}{2}$.

14. $\frac{4\, (17\, b + 3\, d)^2}{5}$.

XXVI.

In the formulas, $r = \frac{a - p}{t\, p}$, and $t = \frac{a - p}{r\, p}$, if p becomes greater than a, r in the first, and t in the second, will become negative, as may be seen by changing the signs of both members: $-r = \frac{p - a}{t\, p}$, and $-t = \frac{p - a}{r\, p}$. In these cases the principal is greater than the amount. Therefore both must be considered as cases of discount. If the rate is negative, it is pretty evidently a case of discount. With regard to the time, if we consider time past as positive, time to come ought to be considered negative. The time being negative, shows in what time a sum equal to a, put at interest now, will amount to the principal; or, in other words, it shows for how long a time a principal p must be discounted at a rate r in order to be worth only a sum a.

When a and p are equal, r in the first and t in the second are equal to zero.

The values of the formulas $\frac{a\,p\,(m+n)}{m\,q-n\,p}$ and $\frac{a\,n\,(p+q)}{m\,q-n\,p}$ can become negative only by means of the denominator, in which alone there is a quantity affected with the sign —. The denominator is the same in both; therefore if one becomes negative, the other will be negative also. They will be negative when $n\,p$ is greater than $m\,q$; thus:

$$n\,p > m\,q:$$

dividing by n and q, it becomes

$$\frac{p}{q} > \frac{m}{n}.$$

Hence it appears that this condition requires the fraction $\frac{p}{q}$ to be greater than $\frac{m}{n}$. Now, by recurring to the conditions of the question, the absurdity will be seen. For, if something be added to the numerator, it will increase the fraction; and if something be added to the denominator, it will diminish the fraction. But the condition requiring $\frac{p}{q}$ to be greater than $\frac{m}{n}$ is exactly the reverse; that is, it requires the numerator to be diminished, in the first instance, in order to furnish a fraction equal to $\frac{m}{n}$; and the denominator diminished, in the second instance, in order to furnish a fraction equal to $\frac{p}{q}$, larger than $\frac{m}{n}$. Hence, if we suppose a to be subtracted, instead of being added, it will answer the conditions.

This will appear in another way, if we suppose that both numerator and denominator have come out negative on account of taking $\frac{p}{q}$ greater than $\frac{m}{n}$. If we go back to the original equation, and introduce these quantities with the sign —, we shall obtain the same result.

Suppose x the numerator, and y the denominator of the fraction required.

By the conditions,

$$\frac{x+a}{y}=\frac{m}{n}$$

$$\frac{x}{y+a}=\frac{p}{q}.$$

But, by our supposition, x and y both come out negative. If we so insert them, they become

$$\frac{-x+a}{-y}=\frac{m}{n},$$

$$\frac{-x}{-y+a}=\frac{p}{q}.$$

Changing the signs of the numerator and denominator in the first members,

$$\frac{x-a}{y}=\frac{m}{n}$$

$$\frac{x}{y-a}=\frac{p}{q}.$$

This shows that a should be subtracted for this condition.

The values of the formulas,

$$\frac{a\,p\,(m+n)}{n\,p-m\,q} \text{ and } \frac{a\,n\,(p+q)}{n\,p-m\,q},$$

become negative when

$$m\,q > n\,p, \text{ which gives}$$

$$\frac{m}{n} > \frac{p}{q}.$$

If we pursue the same course of reasoning as in the last, we shall find that, to answer this condition, a must be added to the numerator and to the denominator, instead of being subtracted from them.

In the next example, let x and y represent the two parts. Then by the conditions,

$$x+y=a$$
$$r\,x+s\,y=b.$$

These, reduced, give

$$x = \frac{b - a\,s}{r - s} \text{ and } y = \frac{a\,r - b}{r - s}.$$

If now we suppose r to be greater than s, the denominators will be positive, and the signs of the values of x and y will depend upon the numerators. Now x will be positive as long as b is greater than $a\,s$, but if $a\,s$ is greater than b, x will be negative.

If we observe the process in the reduction of the equations, we shall see the reason.

$$x + y = a \quad \text{gives}$$
$$y = a - x.$$

Substituting this in the second, it becomes

$$r\,x + s\,a - s\,x = b$$
$$\text{or} \quad (r - s)\,x + a\,s = b.$$

Now it appears that, since $a\,s$ is supposed greater than b, and $r - s$ is a positive quantity, x must be negative in order to render the term $(r - s)\,x$ negative; otherwise the first member would not be equal to the second.

Supposing x negative, the equations become

$$y - x = a$$
$$s\,y - r\,x = b.$$

Reducing these, we have

$$x = \frac{a\,s - b}{r - s}, \text{ and } y = \frac{a\,r - b}{r - s},$$

in which both x and y are positive.

Hence the question for this condition should be enunciated thus: To find two numbers whose difference is a, and if the larger be multiplied by s and the smaller by r, the difference is b.

If we take the other value of y, viz. $\frac{a\,r - b}{r - s}$, and render it negative by supposing b greater than $a\,r$, we shall obtain a similar result. But x instead of y will be the larger, and the larger will be multiplied by r instead of s.

The values of y and x cannot both be rendered negative at the same time, while we suppose r greater than s. For the

supposition that renders $\frac{b - a\,s}{r - s}$ negative, viz. that $a\,s$ is greater than b, necessarily renders the other $\frac{a\,r - b}{r - s}$ positive; for $a\,r$ is greater than $a\,s$; $a\,s - b$ being positive, $a\,r - b$ is so for a stronger reason.

Also the supposition that would render the value of y negative, would necessarily render x positive; for b being greater than $a\,r$ will, for a stronger reason, be greater than $a\,s$, which is smaller than $a\,r$.

If we suppose $a\,s = b$, we have

$$x = \frac{b - a\,s}{r - s} = \frac{b - b}{r - s} = \frac{0}{r - s},$$

or $$x = 0.$$

Also $$y = \frac{a\,r - b}{r - s} = \frac{a\,r - a\,s}{r - s} = \frac{a\,(r - s)}{r - s} = a.$$

Again, if we suppose $a\,r = b$, we have

$$y = \frac{a\,r - b}{r - s} = \frac{a\,r - a\,r}{r - s} = \frac{0}{r - s} = 0.$$

And $$x = \frac{b - a\,s}{r - s} = \frac{a\,r - a\,s}{r - s} = \frac{a\,(r - s)}{r - s} = a.$$

The values of x and y cannot both be zero at the same time; for the supposition which renders one of them zero, necessarily renders the other equal to a.

Hitherto we have supposed the denominator positive. Let us now suppose that s is greater than r, which will render the denominator negative.

If now we suppose b greater than $a\,s$, the value of x will be negative; for the numerator, $b - a\,s$, will be positive, while the denominator is negative. This supposition will render y positive. For $b - a\,s$ being positive, $a\,s - b$ would be negative, and $a\,r$ being less than $a\,s$, $a\,r - b$ would be negative, for a still stronger reason; and both numerator and denominator being negative, the value is positive.

In the same manner, if we suppose $a\,r$ to be greater than b y will be negative and x positive.

Both cannot be negative at once.

In order to see why x ought to be negative in the first supposition, let us recur to the original equations.

$$x + y = a$$
$$r\,x + s\,y = b$$
$$y = a - x$$
$$r\,x + a\,s - s\,x = b$$
$$a\,s + (r - s)\,x = b.$$

Now, by the supposition, $a\,s$ is greater than b; but s being greater than r, $r - s$ is negative; therefore x must be negative also, in order to render the term $(r - s)\,x$ positive or additive.

The negative results obtained in this way must be interpreted precisely the same as the other.

Let us suppose $r = s$.

Then $$x = \frac{b - a\,s}{r - s} = \frac{b - a\,s}{s - s} = \frac{b - a\,s}{0},$$

and $$y = \frac{a\,r - b}{r - s} = \frac{a\,s - b}{s - s} = \frac{a\,s - b}{0}.$$

This supposition renders both x and y infinite; the one an infinite positive, and the other an infinite negative, according as we make $a\,s$ greater or less than b.

Or, more properly speaking, they are both impossible quantities. For, if we recur again to the original equations,

$$x + y = a$$
$$r\,x + s\,y = b$$
$$r\,x + s\,a - s\,x = b$$
$$(r - s)\,x + s\,a = b.$$

Now $r - s$ being zero, it is impossible to substitute any value for x, so as to render $s\,a$ different from b, as our supposition required.

In fact, since $r = s$, if we suppose a to be divided into any number of parts, so that $a = c + d + e +$ &c. we shall necessarily have $s\,a = s\,c + s\,d + s\,e +$ &c.

Hence the case supposed is impossible, as the formula shows, and we must necessarily have $s\,a = b$.

But this supposition renders the numerators zero also; and the formulas become

$$x = \frac{b - a\,s}{r - s} = \frac{a\,s - a\,s}{s - s} = \frac{0}{0}$$

$$y = \frac{a\,r - b}{r - s} = \frac{a\,s - b}{s - s} = \frac{a\,s - a\,s}{s - s} = \frac{0}{0}$$

This shows that both values are indeterminate. This is as it ought to be; for, as we have just seen, it makes no difference into what parts a is divided; for, since all the parts are to be multiplied by s the sum of the products will be $a s$.

XXVII.

1 & 2. See Book.
3. 5 and 4.
4. 6 and $16\frac{1}{2}$.

XXVIII.

1. See Book.
2. 86.
3. 19.
4. 57.
5. 89.
6. 91.

XXIX.

1. See Book.
2. $\frac{12}{25}$.
3. $\frac{73}{20}$.
4. $\frac{123}{23}$.
5. $\frac{7}{9}$ —
6. $\frac{23}{29}$ —
7. $\frac{13}{7}$ —
8. $\frac{62}{15}$ —
9. $\frac{7}{4}$ —
10. $\frac{11}{4}$ —
11. $\frac{81}{4}$ +
12. 5.291 +
13. 15.588 +
14. 164.52 +
15. 15.6004 +
16. 14.61 +
17. 26.839 +

18. 4.1778 +
19. 1.932 +
20. .774 +
21. .81649 +
22. .1247 +
23. .02945 very nearly.

XXX.

1, 2, & 3. See Book.
4. 52.1535 + rods.
5. 108 rods.
6. 25.331 + inches.
7. See Book.
8. A's capital £9$\frac{1}{6}$, B's £120.
9. Length 24 rods, breadth 15.
10. 139.5807 rods, and 74.443 rods.
11. Length 18 feet, breadth 14.
12. Length 30 yards, breadth 25.
13. 12 men, 15 women.
14. 104 rods long, 65 rods wide.
15. 20 feet long, 16 feet wide, and 14 feet high.
16. Length 18 rods, breadth 6.
17. 48 and 3.

XXXI.

1 & 2. See Book.
3. 8.
4. See Book.
5. 71.
6. 18.
7. 98.
8. See Book.
9. Same as 8.
10. 57.
11. 203.

12. 426.
13. 1258.
14. 8007.
15. 60104.

XXXII.

1. $\frac{6}{7}$.
2. $\frac{3}{10}$.
3. $\frac{14}{9}$.
4. $\frac{116}{37}$.

5, 6, & 7. See Book.

8. 2.6085 +
9. 29.523 +
10. 2.85111 +
11. 2.31303 +
12. 1.20257 +
13. .8972 +
14. .28433 +

XXXIII.

1 & 2. See Book.

3. 28.4798 + inches.
4. 12.9074 + inches.
5. 16.2623 + inches.
6. 25.8148 + inches.
7. 8 the greater, 7 the less number.
8. Length 71.1995 inches, breadth 56.9596, depth 28.4798.
9. Length 60.6351 + inches, breadth 45.4763+, depth 30.3175 +
10. Length 84 rods, breadth 9 rods.
11. Length 12 feet, breadth 10, height 10.

XXXIV.

1 to 4 See Book.

5. 16.

6. £30.
7. A had $3.50, B $21.
8. 75 in rank, and 16 in file.
9. 75 sheep, at 16 shillings each.
10. 18 yards of the finer at 20 shillings, and 20 of the coarser at 16 shillings.
11. One 10 yards, the other 26.
12. One 26 feet, the other 38 feet in length.
13. 7 yards of the finer, which cost $$8\frac{1}{6}$. The coarser piece cost $5.
14. £20.
15. 3 inches.
16. A went 117 miles, B went 130.
17. 152 miles, or 56 miles.
18. Length 15 rods, breadth 10. He gave $150 for fencing it.
19. A went 192, and B 128 miles. A travelled 24, and B 16 miles per day.
20. The length must be increased 6.2131 + rods, the breadth 4.9705 +
21. 256 square yards.

XXXV.

1 & 2. See Book.

3. Let x and y represent the two numbers; then

$$y = \frac{a}{2} \pm \frac{(2\,b - a^2)^{\frac{1}{2}}}{2}$$

$$x = \frac{a}{2} \mp \frac{(2\,b - a^2)^{\frac{1}{2}}}{2}.$$

In the first place, we observe, that the values of x and y are the same, except the double sign. Or, in other words, when $x = \frac{a}{2} - \frac{(2\,b - a^2)^{\frac{1}{2}}}{2}$, $y = \frac{a}{2} + \frac{(2\,b - a^2)^{\frac{1}{2}}}{2}$, and vice versa. That is, both are properly given at once. This ought to be so, for they both enter into the original equation alike.

Both will be positive when

$$\frac{a}{2} > \frac{(2\,b - a^2)^{\frac{1}{2}}}{2}$$

which gives $$\frac{a^2}{4} > \frac{2\,b - a^2}{4}$$

$$a^2 > 2\,b - a^2$$

$$2\,a^2 > 2\,b$$

or $$a^2 > b$$

That is, when a^2 is greater than b.

One will be positive and the other negative when

$$\frac{a}{2} < \frac{(2\,b - a^2)^{\frac{1}{2}}}{2}$$

which gives $$\frac{a^2}{4} < \frac{2\,b - a^2}{4}$$

or $$a^2 < 2\,b - a^2$$

$$a^2 < b.$$

Suppose that $$y = \frac{a}{2} - \frac{(2\,b - a^2)^{\frac{1}{2}}}{2}$$ and that

$$a^2 < b.$$

In this case, this value of y is negative, while that of x is positive. If we change the signs of this negative quantity, it becomes positive. Hence

$$-\frac{a}{2} + \frac{(2\,b - a^2)^{\frac{1}{2}}}{2}$$

is a positive quantity.

To see how the negative quantity should be interpreted, let us return to the original equation, and insert y with the negative sign.

The first becomes

$$x - y = a,$$

But the second power of $-y$ is $+y^2$. Hence the second is not altered, but remains

$$x^2 + y^2 = b.$$

If these equations be reduced, we find the first value of y to be $-\frac{a}{2} + \frac{(2b - a^2)^{\frac{1}{2}}}{2}$ which we have just seen to be positive, and the value of x will be the same as before. The values of both are not given at once, as before, because x and y enter differently into the equations.

To answer the above conditions, the question should be proposed thus:

There are two numbers, whose difference is a, and the sum of whose second powers is b. Required the numbers.

When $a^2 = 2b$ the expression $\frac{(2b - a^2)^{\frac{1}{2}}}{2} = 0$, and the values of both y and x become $\frac{a}{2}$.

When $a^2 > 2b$ the quantity $2b - a^2$ becomes negative; consequently the radical quantity $\frac{(2b - a^2)^{\frac{1}{2}}}{2}$ becomes imaginary: therefore this condition is not possible.

4. Let x and y represent the numbers; then

$$x - y = a$$
$$x^2 + y^2 = b.$$

This question answers to the negative value of x and y in the last example.

The values are

$$x = \frac{a}{2} \pm \frac{(2b - a^2)^{\frac{1}{2}}}{2}$$

$$y = -\frac{a}{2} \pm \frac{(2b - a^2)^{\frac{1}{2}}}{2}.$$

The first value of x is necessarily positive. The first value of y will be positive, if $\frac{a}{2}$ is less than $\frac{(2b - a^2)^{\frac{1}{2}}}{2}$; in which case it will answer the conditions of the question. But if $\frac{a}{2}$ is

greater than $\frac{(2b - a^2)^{\frac{1}{2}}}{2}$, y will be negative. If y be inserted into the original equation as a negative quantity, or with a sign contrary to that which it had at first, we have

$$x + y = a$$
$$x^2 + y^2 = b.$$

This is precisely the same as the original equations in the third example.

The second value of y is necessarily negative; but the second value of x may be positive or negative, according as $\frac{a}{2}$ is greater or less than $\frac{(2b - a^2)^{\frac{1}{2}}}{2}$.

If y is negative and x positive, they must be interpreted precisely the same as we have just done for the first values.

If both x and y are negative, we must insert both with opposite signs into the original equations, and they become

$$-x + y = a, \text{ or } y - x = a$$
$$x^2 + y^2 = b.$$

This does not alter the enunciation of the question; it simply makes y take the place of x, and x of y.

If the quantity $\frac{(2b - a^2)^{\frac{1}{2}}}{2}$ becomes zero, as it may, if $2b = a^2$, we shall have

$$x = \frac{a}{2}, \text{ and } y = -\frac{a}{2}.$$

This value of y must be interpreted the same as the negative value of y above.

The values become imaginary in the same manner as in the third example.

5. Let x and y represent the two numbers.

Then $\quad x - y = a,$
$\quad\quad\quad x^3 - y^3 = b.$

$$x = \frac{a}{2} \pm \left(\frac{4b - 3a^3}{12a}\right)^{\frac{1}{2}}$$

$$y = -\frac{a}{2} \pm \left(\frac{4b - 3a^3}{12a}\right)^{\frac{1}{2}}$$

If this result be examined in the manner of the two last examples, it will be found that the first value of x cannot become negative, but the second value may be either positive or negative. The first value of y may be either positive or negative; the second value is necessarily negative.

When x is positive and y negative, if they be so inserted into the original equations, they become

$$x + y = a$$
$$x^3 + y^3 = b;$$

which show that, to answer this condition, the question should be, to take the sum of two numbers to make a, and the sum of the third powers to make b. The sign of y will be changed in the second equation as well as in the first, in this, because the third power of any quantity has the same sign as the quantity itself.

x cannot be negative except when y is negative also. This case will simply change the places of x and y in the original equation, which will then become

$$y - x = a$$
$$y^3 - x^3 = b.$$

These values of x and y may also become imaginary.

6. Let x represent the number of sheep purchased. Then

$$\frac{a}{x} - c = \frac{a}{x + b}$$

$$x = -\frac{b}{2} \pm \left(\frac{a\,b}{c} + \frac{b^2}{4}\right)^{\frac{1}{2}}$$

First we may observe, that these values of x cannot become imaginary, because the quantity within the radical sign cannot become negative.

The first value of x cannot be negative, because the quantity $\left(\frac{a\,b}{c} + \frac{b^2}{4}\right)^{\frac{1}{2}}$ is necessarily greater than $\frac{b}{2}$. For $\left(\frac{b^2}{4}\right)^{\frac{1}{2}}$ alone is equal to $\frac{b}{2}$.

The second value of x is necessarily negative. To find how this must be interpreted, we recur to the original equation, and put in x with the opposite sign :

$$\frac{a}{-x} - c = \frac{a}{-x+b}$$

or $$-\frac{a}{x} - c = -\frac{a}{x-b}.$$

Changing all the signs,

$$\frac{a}{x} + c = \frac{a}{x-b}.$$

This may be expressed thus:

A man bought a number of sheep for a number a of dollars; and on counting them he found, that if he had bought a number b less for the same money, they would have cost a sum c more apiece. How many did he buy?

7. Let x represent the number purchased. Then

$$\frac{a}{x} + d = \frac{c}{x-b}.$$

$$x = -\frac{a+bd-c}{2d} \pm \left(\frac{ab}{d} + \frac{(a+bd-c)^2}{4d}\right)^{\frac{1}{2}}.$$

These values cannot become imaginary, because if $a+bd-c$ were negative, its second power would be positive; therefore the quantity in the radical is necessarily positive.

The first value is necessarily positive, and the second necessarily negative, for the same reason that those in the last example were so.

We introduce x with the opposite sign, as follows:

$$\frac{a}{-x} + d = \frac{c}{-x-b}$$

$$-\frac{a}{x} + d = -\frac{c}{x+b}.$$

Changing all the signs,

$$\frac{a}{x} - d = \frac{c}{x+b},$$

this shows, that, instead of reserving the number b, he added a number b to them, and that he lost a sum d per head by the bargain.

8. This question gives

$$x + \frac{x^2}{100} = a.$$

Hence $\quad x = -50 \pm (100\,a + 2500)^{\frac{1}{2}}.$

The quantity $(100\,a + 2500)^{\frac{1}{2}}$ must necessarily be greater than 50. Therefore the first value of x is necessarily positive, and the second negative.

To find what this negative result signifies, put x into the original equation with the sign —

$$-x + \frac{x^2}{100} = a$$

or $\quad \dfrac{x^2}{100} - x = a.$

This may be expressed thus :

A merchant sold a quantity of brandy for a sum a, which he reckoned at a rate per cent. equal to the cost, and then diminished it by the cost. How much did it cost ?

XXXVI.

1. $243\,a^{10}\,b^{15}\,m^5$.
2. $\dfrac{8\,a^{15}\,c^6}{125\,b^{12}\,d^9}$.
3. $9\,c^2 + 12\,c\,d + 4\,d^2$.
4. $64\,a^3 - 48\,a^2\,b\,c + 12\,a\,b^2\,c^2 - b^3\,c^3$.
5. $a^5 - 5\,a^4\,b + 10\,a^3\,b^2 - 10\,a^2\,b^3 + 5\,a\,b^4 - b^5$.
6. $16\,a^8\,c^4 - 32\,a^6\,c^5 + 24\,a^4\,c^6 - 8\,a^2\,c^7 + c^8$.
7. $(3\,b - c)^8$.
8. $(a - c + 2\,d)^{15}$.
9. $(2\,a^2 - 4\,c^3)^{21}$.
10. $(7\,m + 2\,c)^4$.
11. $2\,a^3 + 18\,a^2\,b + 54\,a\,b^2 + 54\,b^3$.

12. $12\,a^2\,b\,c - 12\,a\,b\,c^2 + 3\,b\,c^3$.

13. $9\,a^3 + 12\,a^2\,b + 4\,a\,b^2 + 27\,a^2\,b^2 - 36\,a\,b\,c^2 + 12\,b^2\,c^2$.

14. $4\,a^6 - 4\,a^5\,b + a^4\,b^2 + 8\,a^4\,b\,c - 8\,a^3\,b^2\,c + 2\,a^2\,b^3\,c + 4\,a^2\,b^2\,c^2 - 4\,a\,b^3\,c^2 + b^4\,c^2$.

15. $3\,a\,b^2$.

16. $-5\,a^{\frac{5}{3}}\,b^2\,c^{\frac{1}{3}}$.

17. $2\,a^2\,x^{\frac{m}{5}}\,r^{\frac{1}{3}}$.

18. $\dfrac{a^{\frac{4}{3}}\,b^{\frac{2}{3}}}{3\,c\,d^2}$.

19. $3\,a^2\left(\dfrac{c^3}{b^2\,m}\right)^{\frac{1}{4}}$.

20. $(2\,m - x)^3$.

21. $(3\,a + x)^{\frac{m}{6}}$.

XXXVII.

1. $a^2 + 2\,a\,x + x^2$.
2. $x^2 - x + \frac{1}{4}$.
3. $2\,x^2 - x + 3$.
4. $x^3 + 2\,x^2 + 3\,x + 4$.

XXXVIII.

1. $x^2 + 2\,x - 4$.
2. $x^2 - 2\,x + 1$.
3. $3\,x - 2\,a$.
4. $2\,x - 1$.

XXXIX.

1. 1593,70224,65957 (437 **Answer.**

$4^5 =$ 1024

1st dividend. 5697 (1280 $= 4^4 \times 5 =$ 1st divisor.

$(43)^5 =$ 146788443

2d dividend. 12581781,6 (17094005 $= (43)^4 \times 5 =$ 2d divisor.

$(437)^5 =$ 15937022465957

2. 364,6915,8961 (437 **Answer.**

$4^4 =$ 256

1st dividend. 1086 (256 $= 4^3 \times 4 =$ 1st divisor.

$(43)^4 =$ 3418801

2d dividend. 2281148 (318028 $= 43^3 \times 4 =$ 2d divisor.

$(437)^4 =$ 36469,158961

3. 481,890304 (28 **Answer.**

$2^6 =$ 64

1st dividend. 4178 (192 $= 2^5 \times 6 =$ 1st divisor.

$(28)^6 =$ 481890304

4. 1349,2928512 (28 **Answer.**

$2^7 =$ 128

1st dividend. 12212 (448 $= 2^6 \times 7$

$(28)^7 =$ 13492928512

XL.

1. See Book.

2. $3\,x^2\,(2\,a\,x)^{\frac{1}{3}}$.

3 & 4. See Book.

5. $\frac{3\,a\,x}{4\,m}\left(\frac{5\,a\,x^2 - 4\,a^4\,x^3\,c^3}{n^2}\right)$

XLIV.

1 & 2. See Book.

3. $a^9 + 9\,a^8\,b + 36\,a^7\,b^2 + 84\,a^6\,b^3 + 126\,a^5\,b^4 + 26\,a^4\,b^5 + 84\,a^3\,b^6 + 36\,a^2\,b^7 + 9\,a\,b^8 + b^9$

4. $m^{13} + 13\,m^{12}\,n + 78\,m^{11}\,n^2 + 286\,m^{10}\,n^3 + 715\,m^9\,n^4 + 1287\,m^8\,n^5 + 1716\,m^7\,n^6 + 1716\,m^6\,n^7 + 1287\,m^5\,n^8 + 715\,m^4\,n^9 + 286\,m^3\,n^{10} + 78\,m^2\,n^{11} + 13\,m\,n^{12} + n^{13}$.

5, 6, & 7. See Book.

8. $m^7 - 7\,m^6\,n + 21\,m^5\,n^2 - 35\,m^4\,n^3 + 35\,m^3\,n^4 - 21\,m^2\,n^5 + 7\,m\,n^6 - n^7$.

9. $16\,a^4 - 32\,a^3\,b\,c^2 + 24\,a^2\,b^2\,c^4 - 8\,a\,b^3\,c^6 + b^4\,c^8$.

10. $a^{15}\,c^5 - 10\,a^{12}\,c^8 + 40\,a^9\,c^9 - 80\,a^6\,c^{14} + 80\,a^3\,c^{17} - 32\,c^{20}$.

11, 12, & 13. See Book.

14. $8\,a^3 - 12\,a^2\,b + 6\,a\,b^2 - b^3 + 12\,a^2\,c^2 - 12\,a\,b\,c^2 + 3\,b^2\,c^2 + 6\,a\,c^4 - 3\,b\,c^4 + c^6$.

15. $2187\,a^{35} - 10206\,a^{32}\,d + 20412\,a^{29}\,d^2 - 22680\,a^{26}\,d^3 + 15120\,a^{23}\,d^4 - 6048\,a^{20}\,d^5 + 1344\,a^{17}\,d^6 - 128\,a^{14}\,d^7$.

16. $2401\,b^8 + 2744\,b^6\,c + 1176\,b^4\,c^2 + 224\,b^2\,c^3 + 16\,c^4 - 1372\,b^6\,d^3 - 588\,b^4\,d^3 - 352\,b^2\,c^2\,d^3 - 32\,c^3\,d^3 + 334\,b^4\,d^6 + 168\,b^2\,c\,d^6 + 24\,c^2\,d^6 - 28\,b^2\,d^9 - 8\,c\,d^9 + d^{12}$.

17. $a^{39} - 26\,a^{36}\,b^2 + 312\,a^{33}\,b^4 - 2288\,a^{30}\,b^6 + 11440\,a^{27}\,b^8 - 41184\,a^{24}\,b^{10} + 109824\,a^{21}\,b^{12} - 219648\,a^{18}\,b^{14} + 329472\,a^{15}\,b^{16} - 366080\,a^{12}\,b^{18} + 262864\,a^9\,b^{20} - 159744\,a^6\,b^{22} + 53248\,a^3\,b^{24} - 8192\,b^{26}$.

18. $a^{10} - 5\,a^8\,c - 10\,a^8\,d + 10\,a^6\,c^2 + 40\,a^6\,c\,d + 40\,a^6\,d^2 - 10\,a^4\,c^3 - 60\,a^4\,c^2\,d - 120\,a^4\,c\,d^2 - 80\,a^4\,d^3 + 5\,a\,c^4 + 40\,a\,c^3\,d + 120\,a\,c^2\,d^2 + 160\,a\,c\,d^3 + 80\,a\,d^4 - c^5 - 10\,c^4\,d - 40\,c^3\,d^2 - 80\,c^2\,d^3 - 80\,c\,d^4 - 32\,d^5$.

19. $a^3 - 6\,a^2\,d + 12\,a\,d^2 - 8\,d^3 + 3\,a^2\,c^2\,d - 12\,a\,c^2\,d^2 + 12\,c^2\,d + 3\,a\,c^4\,d^2 - 6\,c^4\,d^2 + c^6\,d^3$.

20. $a^3 - 3a^2b + 3ab^2 - b^3 + 6a^2c^2 + 12abc^2 - 6b^2c^2 - 3a^2d^3 + 6abd^3 - 3b^2d^3 + 12ac^4 + 12ac^2d^3 + 3ad^6 - 12bc^4 - 12bc^2d^3 - 3bd^6 - 8c^6 - 12c^4d^3 - 6c^2d^6 - d^9$.

21. $16807\,a^{10}b^{25} - 120050\,a^{13}b^{20}c^2 + 343000\,a^{16}b^{15}c^4 - 490000\,a^{19}b^{10}c^6 + 350000\,a^{22}b^5c^8 - 100000\,a^{25}c^{10}$.

XLVI.

1. 300.
2. 30300 yards.
3. 8 hours.
4. They will meet in 204 days. A will travel 20910 miles, B 4080 miles.
5. 8 days and 96 miles, or 15 days and 120 miles.
6. The equation of this question is,

$$\frac{x(1+x)}{2} + \frac{x(42-2x)}{2} = 165$$

which gives

$$x = \pm\frac{23}{2} + \frac{43}{2}$$

$$x = 33 \text{ or } 10.$$

At the end of the tenth day, B, after having travelled 2 miles, would meet A. The next day, B would be stationary. The second day after their meeting, he would travel -2 miles; that is, 2 miles in a direction contrary to that in which he had been going; the third day -4 miles, the fourth -6, &c. A at this time would go much faster than B; but the successive differences of B's daily journeys being double those of A, he would soon go faster than A, and overtake him again, as the equation shows, at the end of the thirty-third day.

7. 15 gallons of brandy, 17 of rum, and 19 of water.
8. 432.
9. The first 5 shillings, the second 7 shillings, and the third 9 shillings per day.

XLVII.

1. See Book
2. $7\frac{1}{2}$.

3. $43\frac{3}{4}$.
4. $237\frac{5}{7}$.
5. $42\frac{3}{4}$.
6. 98415.
7. 1953140.
8. See Book.
9. 8.366 +
10. 2.4494 +

XLVIII.

1. 131,072.
2. 1,048,576.
3. 524,288.
4. 2,048.
5. 2,048.
6. 4.
7. 1,048,576.
8. 262,144.
9. 1,048,576.
10. 512.
11. 64.
12. 32.
13. 16.
14. 8.

LI.

1 to 8. See Book.
9. $x = .885902$.
10. $x = .104584$.
11. $x = .0091979$.
12. See Book.
13. $x = 1.85703$.

LII.

1 to 4. See Book.
5. $241.975.

6. \$346.942.
7. \$183.11.
8. 4.196 + per cent.
9. It will be doubled in 11.8956 years, or 11 years, 10 months, 22 days. It will be tripled in 18.854 years, or 18 years, 10 months, 7 days.
10. 35.047 per cent. for the whole time; 3.05, nearly, per year.
11. 7,165,600.
12. 23.07 years, very nearly.
13. 3.156 per cent.
14. 9,878,930.
15. 22.30323 years.
16. 2.9007 per cent.
17. 11,116,730.
18. 12,824,930.
19. 23.15453 years.
20. 38.42847 years.
21. 57.5938 years from 1820.

LIII.

1. See Book.
2. \$1476.837.
3. \$1169.364.
4. The equation $A = \dfrac{a(1+r)[(1+r)^t - 1]}{r}$ solved with regard to a, gives $a = \dfrac{A r}{(1+r)[(1+r)^t - 1]}$.

In this particular case, $a = \$105.133$.

5. Making t the unknown quantity, the above equation becomes $t = \dfrac{\log.\left(\dfrac{A r}{a(1+r)} + 1\right)}{\log.(1+r)}$ which may be solved in a manner similar to that of example 4, Art. LII. Ans. 10.4064 years.
6. 31275.

LIV.

1. Let A = sum put at interest.

a = sum taken out annually.

The sum a being taken out at the end of the first year, the value of it, at the expiration of the term, would be $a\,(1+r)^{t-1}$. That taken out at the end of the second year would be $a\,(1+r)^{t-2}$, &c. That taken out last would be simply a.

The amount of these several sums is $a\,(1+r)^{t-1} + a\,(1+r)^{t-2} + \ldots\ldots + a\,(1+r) + a$. The sum of which is

$$\frac{a\,[(1+r)^t - (1+r)]}{r} + a.$$

The value of A, when the annuity was to cease, would be $A\,(1+r)^t$. The equation, then, is

$$A\,(1+r)^t = \frac{a\,[(1+r)^t - (1+r)]}{r} + a.$$

$$\text{Whence } A = \frac{a\,[(1+r)^t - (1+r)]}{r\,(1+r)^t} + \frac{a}{(1+r)^t}.$$

Or, reducing to a common denominator, and making the reductions,

$$A = \frac{a\,[(1+r)^t - 1]}{r\,(1+r)^t}.$$

In this example, A = \$97.12.

2. \$5880.13.

3. The amount of the deposit, \$5880.13, in 18 years is \$10010.53, and the amount he would receive is \$7235.06, by Art. LIII.

The gain of the company, then, is \$2775.47.

4. \$2214.41.

5. \$1288.94.

6. 8.82929 years.

7. The formula for this question is the same as that in the questions of Art. LII. Ans. 1,735,800.

8. 44142.

9. In this case, A = 1,735,800 : $1 + r = 1.02543$: and the formula $a = \dfrac{A\,r\,(1+r)^t}{(1+r)^t - 1}$ gives $a = 48041$. Ans.

10. 64335.
11. 70018.

Answers to the Miscellaneous Examples.

1. 67, 62, 58, and 53 miles.
2. 42 shillings.
3. A's wages 6 shillings, and B's 5 shillings.
4. \$103 and \$157.
5. 28 yards.
6. 18 and 28.
7. A had 135 acres, B 297, and C 432.
8. 20 and 24.
9. 45 loads.
10. 18, 30, and 42 yards.
11. 36, 45, 54, and 63 shillings.
12. 2 and 6.
13. 10 and 15.
14. 2 and 10 years.
15. 12 companies.
16. 24 and 15.
17. 20 and 33 yards.
18. A's \$1.75, B's \$1.25.
19. 34 gallons of brandy, 43 of rum.
20. A spends \$1110, B \$990.
21. 32 ships.
22. One $21\frac{7}{15}$, another $3\frac{7}{15}$, and the third $11\frac{8}{17}$.
23. A 266, B 114, and C 370 acres.
24. Principal \$750 at $4\frac{1}{2}$ per cent.
25. 25 and 40 gallons.
26. 60, 90, and 165 yards.
27. 45 received 9 pence, and 5 received 15 pence each.
28. 21 and 28.
29. \$1000.
30. A 180, B 300, and C 350 men.

31. 6 hours, 32 minutes, $43\frac{7}{11}$ seconds.

32. Let x = first digit, y = the second; the number will be $10x + y$. Ans. 48.

33. $\frac{4}{21}$.

34. 13 gallons.

35. One 25, the other 29 yards.

36. 27 yards.

37. $22\frac{2}{3}$ rods.

38. The man worked 9 hours 36 minutes, the boy 7 hours 12 minutes.

39. 16 eagles, 16 quarter eagles, 6 half eagles, and 10 dollars.

40. $2.81975.

41. 48 and 72 yards.

42. 20 hours.

43. A's stock $480, B's $420.

44. 9 rods in width, and 16 in length.

45. A put at interest $320 at 5 per cent, B $480 at 4 per cent.

46. A 10, and B 9 miles per hour.

47. A's capital $$9\frac{1}{6}$, B's $120.

48. At 30 shillings per week.

49. 64, 48, 36, and 27 gallons.

50. A 75, B 58, and C 267 dollars.

51. Mace 10 shillings, and cloves 5 shillings per pound.

52. 10 or 33 days.

53. B in 15, and C in 18 days.

54. 27 yards, at $2 per yard.

55. The fore wheel 4, and the hind wheel 5 yards in circumference.

56. 2, 5, and 13.

57. 275 miles.

58. $40.

59. 6 days, and 26 men remained alive.

60. 10 gallons.

61. A 36, and B 30.
62. 25 yards, at \$8 a yard.
63. Breadth 35 rods, length 47.
64. 4 miles an hour.
65. 18 in one, and 24 in the other.
66. 25 miles from London.
67. A had 81.866 + acres at \$2.443. B had 118.133 + acres at \$1.693.

68. Let x = side of the greater,
y = side of the less.

The equations will be

1. $x^3 y + y^3 x = 820$
2. $x^2 - y^2 = 9.$

Dividing the first by $x\,y$

3. $x^2 + y^2 = \frac{820}{x\,y}.$

Raising the third and second to the second power

4. $x^4 + 2\,x^2 y^2 + y^4 = \frac{\overline{820}^2}{x^2 y^2}$
5. $x^4 - 2\,x^2 y^2 + y^4 = 81.$

Subtracting

6. $4\,x^2 y^2 = \frac{820}{x^2 y^2} - 81.$

$$4\,x^4 y^4 = \overline{820}^2 - 81\,x^2 y^2$$

$$x^4 y^4 = \overline{410}^2 - \frac{81}{4}\,x^2 y^2$$

$$x^4 y^4 + \frac{81}{4}\,x^2 y^2 = \overline{410}^2$$

$$x^4 y^4 + \frac{81}{4}\,x^2 y^2 + \left(\frac{81}{8}\right)^2 = \overline{410}^2 + \left(\frac{81}{8}\right)^2$$

$$x^2 y^2 = -\frac{81}{8} \pm \left(\overline{410}^2 + \left(\frac{81}{8}\right)^2\right)^{\frac{1}{2}}$$

$$x^2 y^2 = -\frac{81}{8} \pm \overline{10764961}^{\frac{1}{2}}$$

$$x^2 y^2 = 400$$

$$x\,y = 20.$$

Substituting this value of $x\,y$ in the third, it becomes

$$x^2 + y^2 = 41$$

$$2. \quad x^2 - y^2 = 9$$

adding $2\,x^2 = 50$

$$x^2 = 25$$

$$x = 5.$$

$$y = 4.$$

Having the dimensions of the stacks, the prices of each are readily found. Ans. £25, and £16.

69. A's money was \$50, and B's was continued in trade one month.

END OF THE KEY.